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MARKOVIAN TIME-DELAY SAMPLING POLICIES.(U)
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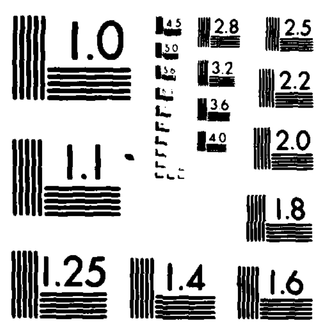

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MARKOVIAN TIME-DELAY SAMPLING POLICIES

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0. Abstract. Sampling procedures are considered for monitoring the output of a production process. Sampling rates are allowed to depend on one of several levels of acceptance or rejection, sampling less frequently when the process is in control. The problem is formulated as a simple Markov process whose properties yield the expected values and the variances of the number of samples and the numbers of various types of adjustments to the process. Computations are given in support of the economic design of variable sampling policies of this type.

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1. Introduction. Two major facets of statistical quality control are acceptance sampling plans for inspecting lots of manufactured items and statistical control charts for monitoring the output of a production process. Central to the present study is the notion that less intensive sampling is needed when consecutive lots of high quality are produced or when the production process continues to remain in control. This idea is not new. Sampling plans for the percent defective were devised along these lines by Dodge (1943), Wald and Wolfowitz (1945), and Dodge and Torrey (1951), who gave rules for switching from 100% sampling to sampling a fraction f. Multi-level inspection plans having sampling fractions in geometric progression were studied by Lieberman and Solomon (1955) and by Derman, Littauer and Solomon (1957) using various rules for shifting to tightened inspection levels. These ideas carry over to monitoring production processes. In the related problem of monitoring stream pollution, Arnold (1970) developed variable sampling policies with subsequent sampling rates depending on the current level of pollution. Unlike process control, this model makes no provision for initiating corrective action. These policies were studied further by Crigler (1973) and by Smeach and Jernigan (1977). The foregoing procedures, all adaptive in that subsequent sampling rates depend on current outcomes, effect greater economy as the quality consistently improves.

Data-dependent sampling strategies may be described stochastically. Lieberman and Solomon (1955) thereby expressed the average fraction inspected in their multi-level plan in terms of the steady-state probabilities of a Markov chain. Similarly, Arnold (1970) found the mean and Smeach and Jernigan (1977) the variance of the sample size in Arnold's variable sampling procedure.

In this study we develop adaptive sampling policies for use with control charts for monitoring the output of a production process. We go beyond known

variable sampling models, allowing for various levels of acceptance and warning and providing for various types of rejection. One of several types of adjustments is initiated when the process is not in control. These features enable us to extend the conventional usage of control charts to include diagnostic capabilities in multiparameter cases. An example is cited. Following Arnold (1970), we achieve variable sampling rates by varying the time delay between successive inspections. On identifying our model as an aperiodic recurrent Markov chain, we obtain the expected values of the sample size and the numbers of various types of adjustments to production. Further studies yield exact and approximate expressions for the variances of these quantities. Attention then is given to the design of time-delay sampling policies.

2. The Model and Assumptions. Let $\underset{\sim}{Y}(t\times 1)$ be the typical quality characteristics of a production process, taking values in the t-dimensional Euclidean space R^t according to a probability measure $P(\cdot)$. We partition R^t into regions $\{I_1, \ldots, I_k, R_1, \ldots, R_m\}$ consisting of outcomes of two types. Outcomes in $\{I_1, \ldots, I_k\}$ indicate different levels of acceptance and warning; the outcomes in $\{R_1, \ldots, R_m\}$ indicate different types of rejections requiring different types of adjustments to production when the process is not in control.

To fix ideas for the case $k = 2$ and $m = 1$, consider the use of Hotelling's T^2 chart (cf. Hotelling (1947)) for monitoring the means of a bicharacteristic process when the process dispersion parameters are known. The regions $\{I_1, I_2, R\}$ may be constructed as in Figure 1 with I_1 an acceptance region, I_2 a warning region, and R a rejection region signalling that the process is not in control and that corrective action is needed. An example having $k = 1$ and $m = 3$ arises in monitoring the means of a bicharacteristic process against one-sided upper alternatives, the object being to diagnose which variables require

adjustment when the process is not in control. The regions $\{I, R_1, R_2, R_3\}$ as in Figure 2 may be chosen with I an acceptance region, R_1 a rejection region requiring adjustment to μ_1 but not μ_2, R_2 a region requiring adjustment to μ_2 but not μ_1, and R_3 a region requiring adjustments to both μ_1 and μ_2.

We henceforth assume that items yielding the characteristics $\underset{\sim}{Y}$ are produced in sequence, one unit of time having elapsed in the production of each. Let $\{d_1, \ldots, d_k\}$ be positive delay parameters not exceeding some maximum delay Δ, and suppose for each $j = 1, 2, \ldots, m$ that adjustments originating in the region R_j require A_j time units for completion. Our basic sampling plan is the following.

<u>Time-Delay Inspection Plan</u>. If $\underset{\sim}{Y} \in I_i$, delay d_i time units before inspecting the next item. If $\underset{\sim}{Y} \in R_j$, begin adjustment as appropriate, delay sampling for A_j time units, and then resume inspection of the adjusted process.

Note that the basic sampling plan is advantageous when inspection costs are high and it is more economical to forego further inspection during adjustment and to sell at reduced prices the items of questionable quality. A modification of the basic plan is to inspect all items consecutively for $A_j - 1$ time intervals during adjustment, and then to resume inspection of the adjusted process. This modification may be appropriate when inspections are critical, ensuring as it does that all items of questionable quality will have been inspected.

In order to study the stochastic behavior of our sampling policies efficiently, we approach the problem as an application of the theory of discrete Markov chains. The latter are characterized in terms of their possible states and the matrix of one-step transition probabilities. This program is carried out in the following sections, first for a single rejection model, then for

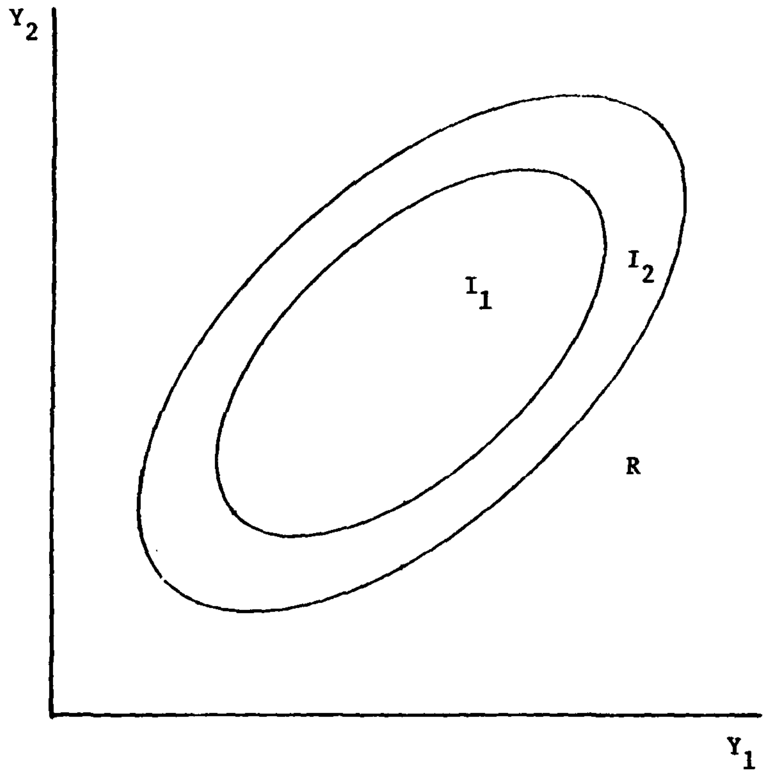

FIGURE 1. Acceptance (I_1), warning (I_2) and rejection (R) regions in monitoring the means of a bicharacteristic production process.

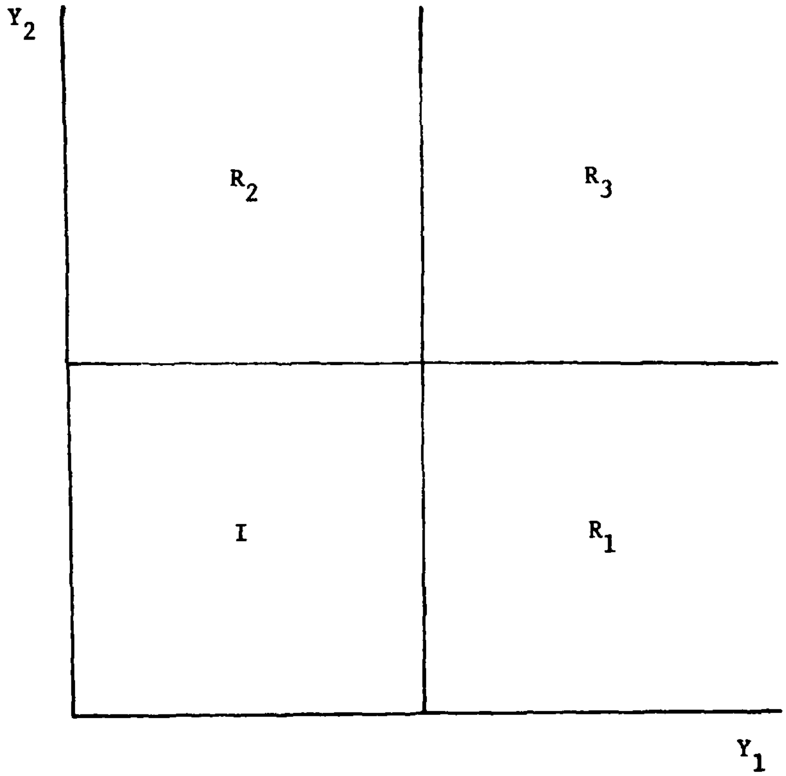

FIGURE 2. Control limits for monitoring the means of a bicharacteristic process against one-sided alternatives with diagnostic capabilities.

the general case. The findings are applied to the design of variable sampling plans of the types considered here.

3. Single Rejection Models. In sequence we identify the Markovian states; we give the one-step transition probabilities in terms of the natural parameters of the problem; and we develop some of the stochastic properties of our time-delay monitoring scheme. Proofs are given in detail in preparation for the sketches outlined in the section following.

Consider the regions $\{I_1, \ldots, I_k, R\}$ with respective probabilities $\{p_1, \ldots, p_k\ q\}$ under $P(\cdot)$ such that $p_1 + \ldots + p_k + q = 1$. An outcome is said to be acceptable if $\underset{\sim}{Y} \in I_i$ for some i, and to be unacceptable if $\underset{\sim}{Y} \in R$, in which case adjustments requiring A time units are initiated. Let E_i be the sampling state that the most recent sample was taken i time units previously and $\underset{\sim}{Y}$ was found acceptable, where $1 \leq i \leq \delta$ and $\delta = \max\{d_1, \ldots, d_k\}$, and let $E_{i,t}$ denote the event of being in sampling state E_i at some nonnegative time t. Similarly let F_j be the sampling state that $\underset{\sim}{Y}$ was found unacceptable and adjustment commenced j time units previously, where $1 \leq j \leq A$. Clearly $\{E_1, \ldots, E_\delta, F_1, \ldots, F_A\}$ are the exclusive and exhaustive states of our time-delay sampling scheme. The procedure is initiated by taking a sample at time t = 0 when the process is in control, the sampling procedure being in state E_1 at time t = 1. It is important to note that the sampling state occupied at time unit t is determined before any sample is taken and before any adjustments to the process have begun.

The matrix of one-step transition probabilities is found in terms of the parameters of the problem as follows. Retain the order $\{E_1, \ldots, E_\delta, F_1, \ldots, F_A\}$, and consider the corresponding stochastic matrix $\underset{\sim}{P} = [P_{rs}]$ of order $(\delta+A) \times (\delta+A)$. For $1 \leq t \leq \delta$ let π_t be the probability of delaying exactly t time units before the next inspection given that the most recent inspection

is acceptable. Because inspection is delayed exactly d_i time units when $\underset{\sim}{Y} \varepsilon I_i$ and this occurs with probability p_i, it follows that

$$\pi_t = \begin{cases} p_i \text{ if } t = d_i \text{ for some } i = 1, 2, \ldots, k \\ 0 \quad \text{otherwise.} \end{cases} \tag{3.1}$$

In one time unit the only possible state transitions are (i) E_i to E_{i+1}, $i = 1, 2, \ldots, \delta-1$; (ii) E_i to E_1, $i = 1, 2, \ldots, \delta$; (iii) E_i to F_1, $i = 1, 2, \ldots, \delta$; (iv) F_j to F_{j+1}, $j = 1, 2, \ldots, A-1$; (v) F_A to E_1; and (vi) F_A to F_1. The only nonzero elements of $\underset{\sim}{P}$ accordingly are those corresponding to these six state transition types. The structure of $\underset{\sim}{P}$ is reflected in Table 1; its typical elements as functions of $\{\pi_1, \ldots, \pi_\delta, q\}$ are considered next on a case-by-case basis. To illustrate the types of arguments used note that

$$\begin{aligned} P_{i,i+1} &= P(E_{i+1,t+1} | E_{i,t}) \\ &= 1 - P(\text{inspect at the next time unit} | E_{i,t}) \\ &= 1 - \frac{P(\text{inspection delays exactly } i \text{ time units})}{P(\text{inspection delays at least } i \text{ time units})} \,. \end{aligned}$$

Now let $\pi(r) = (\pi_r + \pi_{r+1} + \ldots + \pi_\delta)$ be the probability that inspection is delayed at least r time units following a favorable inspection. Routine arguments are used to establish the following.

Case (i). $P_{i,i+1} = 1 - \pi_i/\pi(i)$, $i = 1, 2, \ldots, \delta-1$;

Case (ii). $P_{i,1} = (1-q)\pi_i/\pi(i)$, $i = 1, 2, \ldots, \delta$;

Case (iii). $P_{i,\delta+1} = q\pi_i/\pi(i)$, $i = 1, 2, \ldots, \delta$;

Case (iv). $P_{\delta+i,\delta+i+1} = 1$, $i = 1, 2, \ldots, A-1$;

Case (v). $P_{\delta+A,1} = 1-q$;

Case (vi). $P_{\delta+A,\delta+1} = q$;

Case (vii). $P_{ij} = 0$, otherwise.

TABLE 1. The one-step transition probabilities P_{rs} from state r to state s in a time-delay sampling scheme with single rejection region.

State r	State s									
	E_1	E_2	E_3	$\cdots$	E_δ	F_1	F_2	F_3	$\cdots$	F_A
E_1	P_{11}	P_{12}	0	$\cdots$	0	$P_{1,\delta+1}$	0	0	$\cdots$	0
E_2	P_{21}	0	P_{23}		0	$P_{2,\delta+1}$	0	0	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
$E_{\delta-1}$	$P_{\delta-1,1}$	0	0	$\cdots$	$P_{\delta-1,\delta}$	$P_{\delta-1,\delta+1}$	0	0	$\cdots$	0
E_δ	$P_{\delta,1}$	0	0	$\cdots$	0	$P_{\delta,\delta+1}$	0	0	$\cdots$	0
F_1	0	0	0	$\cdots$	0	0	1	0	$\cdots$	0
F_2	0	0	0	$\cdots$	0	0	0	1	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
F_{A-1}	0	0	0	$\cdots$	0	0	0	0	$\cdots$	1
F_A	$P_{\delta+A,1}$	0	0	$\cdots$	0	$P_{\delta+A,\delta+1}$	0	0	$\cdots$	0

It is seen from the one-step transition matrix $\underset{\sim}{P}$ that the Markov process is aperiodic and recurrent. Its properties yield insight into the stochastic behavior of the number of inspections and the number of adjustments to the process in monitoring over a period of length T. It is readily seen that the number L_T of acceptable inspections in a time interval (0,T] is equal to the number of visits to state E_1 in that interval. Similarly the number M_T of unacceptable inspections is equal to the number of visits to state F_1 in the interval (0,T], and this is precisely the number of adjustments. Let $P_{ij}^{(n)}$ be the (i,j) element of the nth power of $\underset{\sim}{P}$. Then $P_{11}^{(t)}$ is the probability of a favorable inspection at time t, i.e., the probability of transition from E_1 to E_1 in t steps. Similarly $P_{1,\delta+1}^{(t)}$ is the probability that adjustment is initiated at time unit t, i.e., the probability of transition from E_1 to F_1 in t steps. It follows using standard arguments (cf. Arnold (1970), for example) that the expectations are

$$E(L_T) = \sum_{t=1}^{T} P_{11}^{(t)} \tag{3.2}$$

$$E(M_T) = \sum_{t=1}^{T} P_{1,\delta+1}^{(t)} . \tag{3.3}$$

Let N_T be the total number of inspections carried out in the interval (0,T]. The mean $E(N_T)$ and the variances $V(L_T)$ and $V(N_T)$ are given in the following.

<u>THEOREM 3.1</u>. Let L_T and N_T respectively be the number of acceptable and the total number of inspections in monitoring production over the interval (0,T] using time-delay sampling policies. Then the mean and variance of N_T are given respectively by

$$E(N_T) = \sum_{t=1}^{T} P_{11}^{(t)} + \sum_{t=1}^{T} P_{1,\delta+1}^{(t)} \tag{3.4}$$

$$V(N_T) = \sum_{i=1}^{T} \sum_{j=1}^{T} [a_{\min(i,j)}\, a^*_{|i-j|} - a_i a_j] \tag{3.5}$$

where $a_t = P_{11}^{(t)} + P_{1,\delta+1}^{(t)}$ and $a_t^* = (1-q)a_t + q(P_{\delta+1,1}^{(t)} + P_{\delta+1,\delta+1}^{(t)})$. The mean and variance of L_T are given respectively by (3.2) and

$$V(L_T) = \sum_{i=1}^{T} \sum_{j=1}^{T} [b_{\min(i,j)}\, b_{|i-j|} - b_i b_j] \tag{3.6}$$

where $b_t = P_{11}^{(t)}$.

<u>Proof</u>. Expression (3.4) follows directly from (3.2) and (3.3) and the relation $N_T = L_T + M_T$. Expression (3.6) follows in the manner of Smeach and Jernigan (1977) starting with b_t, the probability of an acceptable inspection. To verify (3.5), note that $a_t = P_{11}^{(t)} + P_{1,\delta+1}^{(t)}$ is the probability that an inspection is made at time t. Following Smeach and Jernigan (1977), let Z_t be the indicator of this event and note that $N_T = Z_1 + \ldots + Z_T$. Using the fact that the covariance of Z_i and Z_j may be expressed as $\mathrm{Cov}(Z_i, Z_j) = P(Z_i=1,\ Z_j=1) - E(Z_i)\ E(Z_j)$, we have

$$V(N_T) = \sum_{i=1}^{T} \sum_{j=1}^{T} [P(Z_i=1,\ Z_j=1) - E(Z_i)\ E(Z_j)]\ . \tag{3.7}$$

Let $r = \min(i,j)$ and $s = |i-j|$ and consider the conditional probability $P(Z_{r+s}=1 | Z_r=1)$. Given that $Z_r = 1$, an inspection was made at time $t = r$ and thus the process is either in sampling state E_1 at $t = r + 1$ or in sampling state F_1 at $t = r + 1$, these events occurring with probabilities $(1-q)$ and q, respectively. The law of total probability asserts that

$$\begin{aligned} P(Z_{r+s}=1|Z_r=1) &= P(\text{inspect at } t = r + s|E_{1,r+1})\ P(E_{1,r+1}) \\ &\quad + P(\text{inspect at } t = r + s|F_1(r+1))\ P(F_1(r+1)) \\ &= (P_{11}^{(s)} + P_{1,\delta+1}^{(s)})\ (1-q) + (P_{\delta+1,1}^{(s)} + P_{\delta+1,\delta+1}^{(s)})q \end{aligned} \tag{3.8}$$

where $F_j(t)$ is the event that the process is in sampling state F_j at time t. On writing

$$P(Z_i=1,\ Z_j=1) = P(Z_{r+s}=1|Z_r=1)\ P(Z_r=1) \tag{3.9}$$

and noting that $P(Z_r=1) = E(Z_r) = P_{11}^{(r)} + P_{1,\delta+1}^{(r)}$, we combine (3.9) and (3.8) with (3.7) to get (3.5).

Suppose the inspection plan is modified so that, for any adjustment initiated in (0,T], the next A items are inspected during adjustment. Let N_T^* be the total number of inspections in (0,T] under this scheme. Then (3.4) is modified to

$$E(N_T^*) = \sum_{t=1}^{T} P_{11}^{(t)} + A \sum_{t=1}^{T} P_{1,\delta+1}^{(t)} . \tag{3.10}$$

In a manner similar to our treatment of L_T and N_T, we may investigate further stochastic properties of the number M_T of adjustments to the process. For ease of reference our main results are summarized in the following.

<u>THEOREM 3.2</u>. Let M_T be the number of adjustments in monitoring production over the interval (0,T] using time-delay sampling policies. Then $E(M_T)$ and $V(M_T)$ are given respectively by (3.3) and

$$V(M_T) = \sum_{i=1}^{T} \sum_{j=1}^{T} [c_{\min(i,j)} c^*_{|i-j|} - c_i c_j] \tag{3.11}$$

where $c_t = P_{1,\delta+1}^{(t)}$ and $c_T^* = P_{\delta+1,\delta+1}^{(t)}$.

<u>Proof</u>. Note that $c_t = P_{1,\delta+1}^{(t)}$ is the probability that an adjustment is initiated at time t. Let Z_t be the indicator for this event and note that $M_T = Z_1 + \ldots + Z_T$. The proof now parallels that of Theorem 3.1 on interpreting the conditional probability $P(Z_{r+s}=1|Z_r=1)$ suitably. Given that $Z_r=1$, the process was adjusted at time $t = r$ and thus is in sampling state F_1 at $t = r + 1$. In order that $Z_{r+s} = 1$, adjustment must be initiated at $t = r + s$, assuring that the process is in sampling state F_1 at $t = r + s + 1$. Given that $Z_r = 1$, the event $Z_{r+s} = 1$ thus is attained through the transition from sampling state F_1 to state F_1 in exactly s steps,

i.e., $P(Z_{r+s}=1|Z_r=1) = P^{(s)}_{\delta+1,\delta+1}$. Because $P(Z_r=1) = E(Z_r) = P^{(r)}_{1,\delta+1}$, the expression corresponding to (3.9) is

$$P(Z_i=1,\ Z_j=1) = [P^{(s)}_{\delta+1,\delta+1}]\ [P^{(r)}_{1,\delta+1}] \ . \tag{3.12}$$

The remainder of the proof is identical to that of Theorem 3.1.

The stochastic properties of the number of inspections and the number of adjustments, as set forth in Theorems 3.1 and 3.2, assume a vital role in comparing alternative time-delay sampling plans. We return to this topic in Section 5.

4. Multiple Rejection Models. Motivated in part by the need for diagnostic capabilities mentioned with reference to Figure 2, we consider the case of multiple rejection regions requiring different types of adjustments to the process. An extension of the arguments of the preceding section yields stochastic properties of the number of samples and the numbers of the several types of adjustments in the time interval $(0,T]$.

Consider the regions $\{I_1, \ldots ,I_k, R_1, \ldots ,R_m\}$ with respective probabilities $\{p_1, \ldots ,p_k, q_1, \ldots ,q_m\}$ under $P(\cdot)$ such that $p_1 + \ldots + p_k + q_1 + \ldots + q_m = 1$. An outcome is said to be acceptable if $\underset{\sim}{Y} \in I_i$ for some $i = 1, 2, \ldots ,k$, and to be unacceptable of type j if $\underset{\sim}{Y} \in R_j$, in which case the process is said to be in rejection mode j for A_j time units while the process is undergoing adjustment. Define the sampling states $\{E_1, \ldots ,E_\delta\}$ as before, and let F_{ij} be the sampling state that the process is currently in rejection mode i, having entered that mode j time units previously, for $j = 1, 2, \ldots ,A_i$ and $i = 1, 2, \ldots ,m$. Consider the states $\{E_1, \ldots ,E_\delta, F_{11}, \ldots ,F_{1A_1}, \ldots ,F_{m1}, \ldots ,F_{mA_m}\}$ in that order; these clearly are the exclusive and exhaustive states of our time-delay sampling policies with multiple rejection regions. Let $P = [P_{ij}]$ be the

corresponding matrix of one-step transition probabilities; its structure and typical elements are found as in Section 3.

In one time unit there are six possible state transitions, namely, (i) E_i to E_{i+1}, $i = 1, 2, \ldots, \delta-1$; (ii) E_i to E_1, $i = 1, 2, \ldots, \delta$; (iii) E_i to F_{j1}, $i = 1, 2, \ldots, \delta$, $j = 1, 2, \ldots, m$; (iv) F_{ij} to $F_{i,j+1}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, A_i-1$; (v) F_{iA_i} to E_1, $i = 1, 2, \ldots, m$; and (vi) F_{iA_i} to F_{j1}, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, m$. Thus the only nonzero elements of $\underset{\sim}{P}$ correspond to transitions of these types.

To evaluate typical elements of $\underset{\sim}{P}$ let $R = R_1 \cup \ldots \cup R_m$ and $q = q_1 + \ldots + q_m$, and define $\{A(r) = A_0 + A_1 + \ldots + A_r;\ r = 0, 1, \ldots, m\}$ with $A_0 = 0$. Recalling that $\pi(r) = (\pi_r + \pi_{r+1} + \ldots + \pi_\delta)$, we proceed as in Section 3 to evaluate P_{ij} in the following typical cases.

Case (i). $P_{i,i+1} = 1 - \pi_i/\pi(i)$, $i = 1, 2, \ldots, \delta-1$;

Case (ii). $P_{i,1} = (1-q)\ \pi_i/\pi(i)$, $i = 1, 2, \ldots, \delta$;

Case (iii). $P_{i,\delta+A(j-1)+1} = q_j\ \pi_i/\pi(i)$, $i = 1, 2, \ldots, \delta$, $j = 1, 2, \ldots, m$;

Case (iv). $P_{\delta+A(i-1)+j,\delta+A(i-1)+j+1} = 1$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, A_i-1$;

Case (v). $P_{\delta+A(i),1} = 1-q$, $i = 1, 2, \ldots, m$;

Case (vi). $P_{\delta+A(i),\delta+A(j-1)+1} = q_j$, $k = 1, 2, \ldots, m$, $j = 1, 2, \ldots, m$;

Case (vii). $P_{ij} = 0$, otherwise.

Let L_T be the number of acceptable inspections in the time interval $(0,T]$ and let $\{M_{1T}, M_{2T}, \ldots, M_{mT}\}$ respectively be the number of adjustments of types $1, 2, \ldots, m$. Clearly the number of adjustments of type j initiated during the period is equal to the number of entries into rejection mode j, and this is precisely the number of visits of the process to sampling state F_{j1} in $(0,T]$. Because the process is in sampling state E_1 at time $t = 1$, it follows as before that the probability of a favorable inspection at time t is $P_{11}^{(t)}$. Similarly

$P^{(t)}_{1,\delta+A(j-1)+1}$ is the probability that $\underset{\sim}{Y}$ is unacceptable of type j at time t and thus that an adjustment of type j is initiated. Expressions (3.2) and (3.6) for $E(L_T)$ and $V(L_T)$ continue to hold with $b_t = P^{(t)}_{11}$ defined in terms of the current version of $\underset{\sim}{P}^{(n)}$. Using familiar arguments we have

$$E(M_{jT}) = \sum_{t=1}^{T} P^{(t)}_{1,\delta+A(j-1)+1} \; ; \; j = 1,\, 2,\, \ldots\, ,m. \tag{4.1}$$

Low order moments of the distribution of the total number of inspections are given in the following.

<u>THEOREM 4.1</u>. Let N_T be the total number of inspections in *monitoring production* over the interval (0,T] using time-delay sampling with multiple rejection capabilities. Then the mean and variance of N_T are given respectively by

$$E(N_T) = \sum_{t=1}^{T} P^{(t)}_{11} + \sum_{j=1}^{m} \sum_{t=1}^{T} P^{(t)}_{1,\delta+A(j-1)+1} \tag{4.2}$$

$$V(N_T) = \sum_{i=1}^{T} \sum_{j=1}^{T} [a_{\min(i,j)}\, a^*_{|i-j|} - a_i a_j] \tag{4.3}$$

where $a_t = P^{(t)}_{11} + \sum_{j=1}^{m} P^{(t)}_{1,\delta+A(j-1)+1}$ and

$$a^*_t = (1-q)a_t + \sum_{j=1}^{m} q_j [P^{(t)}_{\delta+A(j-1)+1,1} + \sum_{k=1}^{m} P^{(t)}_{\delta+A(j-1)+1,\delta+A(k-1)+1}] \ . \tag{4.4}$$

<u>Proof</u>. Expression (4.2) follows directly from (3.2) and (4.1) in view of the relation $N_T = L_T + M_{1T} + \ldots + M_{mT}$. To validate expression (4.3), let Z_t be an indicator for the event that an inspection is made at time t and note that $E(Z_t) = a_t$. Our proof parallels that of Theorem 3.1, requiring the conditional probability $P(Z_{r+s}=1|Z_r=1)$. Given that $Z_r=1$, the event $Z_{r+s}=1$ requires transition from one of the states $\{E_1, F_{11}, \ldots ,F_{m1}\}$ at time $t = r + 1$ to one of those same states at $t = r + s + 1$. An extension of our earlier arguments yields

$$P(Z_{r+s}=1|Z_r=1) = P(\text{inspect at } t=r+s|E_{1,r+1})P(E_{1,r+1}) + \sum_{j=1}^{m} P(\text{inspect at } t=r+s|F_{j1}(r+1))P(F_{j1}(r+1)) \tag{4.5}$$

where $F_{j1}(t)$ is the event that the process is in sampling state F_{j1} at time t. On evaluating (4.5) we get (4.4). The remainder of the proof is identical to the proofs of Theorems 3.1 and 3.2.

The remainder of this section is concerned with the number of adjustments of type j undertaken while production is monitored over the interval (0,T]. For ease of reference the results are summarized as follows.

THEOREM 4.2. For j = 1, 2, ... ,m let M_{jT} be the number of adjustments of type j in monitoring production over the interval (0,T] using time-delay sampling with multiple rejection capabilities. Then the mean and variance of M_{jT} are given respectively by (4.1) and

$$V(M_{jT}) = \sum_{u=1}^{T} \sum_{v=1}^{T} [b^j_{\min(u,v)} c^j_{|u-v|} - b^j_u b^j_v] \tag{4.6}$$

where $b^j_t = P^{(t)}_{1,\delta+A(j-1)+1}$ and $c^j_t = P^{(t)}_{\delta+A(j-1)+1,\delta+A(j-1)+1}$.

Proof. The proof parallels that of Theorem 3.2 with Z_t an indicator for the event that a type j adjustment begins at time t. The essential difference lies in evaluating the conditional probability $P(Z_{r+s}=1|Z_r=1)$. However, given that $Z_r=1$, the process is in sampling state F_{j1} at time t = r + 1, and the event $Z_{r+s}=1$ implies the sampling state F_{j1} at t = r + s + 1. The probability of transition from state F_{j1} to state F_{j1} in exactly s steps thus gives

$$P(Z_{r+s}=1|Z_r=1) = P^{(s)}_{\delta+A(j-1)+1,\delta+A(j-1)+1} \,. \tag{4.7}$$

The remainder of the proof is identical to that of Theorem 3.2.

5. Choice of Sampling Plans. The developments of the foregoing sections bear heavily on the choice of adaptive sampling plans for monitoring production. We assume that the types and number of regions may be chosen in the context of each particular problem, but we caution that the remaining parameters may be excessive in number. These include the probabilities $\{p_1, \ldots, p_k, q_1, \ldots, q_m\}$, determined through choice of the boundaries of the regions, and the delay parameters $\{d_1, \ldots, d_k\}$. A detailed treatment of economic models and optimal sampling plans is beyond the scope of this study; various search algorithms may be put to this purpose. Instead we consider more elementary aspects of the choice of sampling plans, illustrating these numerically in selected cases.

Arnold (1970) and Crigler (1973) focused mainly on the expected number of inspections in choosing among the time-delay sampling policies considered there. However, Smeach and Jernigan (1977) emphasized also the importance of the variance of the number of inspections in making that choice. In the model under current study, where adjustments to production may be considerably more expensive than routine inspections, it appears desirable to consider the mean and variance of the number of adjustments as well.

To fix ideas consider again monitoring a bicharacteristic production process using Hotelling's T^2 chart with acceptance (I_1), warning (I_2), and rejection (R) regions as in Figure 1 such that $p_1 = 0.65$, $p_2 = 0.30$, and $q = 0.05$. We retain these values and vary the design through choice of the delay parameters $\{d_1, d_2\}$ subject to $d_1 > d_2$ and a maximum allowable delay of $\Delta = 10$ time units. It is assumed that adjustment of the process requires $A = 5$ time units, and that the process is to be monitored for $T = 1000$ time units. Eight different sampling plans are listed in Table 2 according to $\{d_1, d_2\}$, together with the values of $E(N_T)$ and $E(M_T)$ determined as in Section 3. Suppose budgetary allocations will support a sampling plan averaging between 155 and 160 inspections. The candidates

TABLE 2. Expected numbers of inspections (N_T) and adjustments (M_T) under alternative plans in a time-delay sampling model having regions $\{I_1, I_2, R\}$ with probabilities $\{0.65, 0.30, 0.05\}$, operating over the time interval $(0, 1000]$ with $A = 5$.

Plan	d_1	d_2	$E[N_T]$	$E[M_T]$
I	2	1	540.45	27.02
II	9	1	155.97	7.80
III	3	2	356.84	17.84
IV	8	3	157.10	7.86
V	8	4	149.97	7.50
VI	7	5	158.30	7.92
VII	9	6	126.14	6.31
VIII	10	9	105.35	5.27

are Plans II, IV, and VI, and these accordingly are examined in greater detail in Table 3 with regard to the means and variances of both N_T, the number of inspections, and M_T, the number of adjustments to production. Computations were carried out as prescribed in Section 3. As the means and variances of M_T are comparable, we distinguish among the three plans on the basis of N_T, noting that its variance is considerably smaller for Plan VI. On these grounds Plan VI accordingly would be recommended.

We conclude this section with notes on computations. For T as large as 1000 the computations of means and variances are formidable. In addition to establishing exact variance formulas as noted earlier, Smeach and Jernigan (1977) gave approximations to the mean and variance of N_T in the model considered there. Their approximations carry over to the present study in the following manner.

Because the Markovian sampling scheme of Section 4 is aperiodic and recurrent, the limiting probabilities for the several states are given by

$$p_1^{(\infty)} = [1 + \sum_{i=1}^{\gamma-1} (\prod_{j=1}^{i} P_{j,j+1})]^{-1} \tag{5.1}$$

$$p_n^{(\infty)} = (\prod_{j=1}^{n-1} P_{j,j+1})\, p_1^{(\infty)}, \quad n = 2, 3, \ldots, \gamma \tag{5.2}$$

where $\gamma = \delta + A(m)$. Then the limiting probability a_∞ of making an inspection is given by

$$a_\infty = p_1^{(\infty)} + \sum_{j=1}^{m} p_{\delta+A(j-1)+1}^{(\infty)} \, . \tag{5.3}$$

If the convergence of a_t to a_∞ is rapid, then instead of (4.2) one may use the approximation

$$E(N_T) \doteq Ta_\infty \quad . \tag{5.4}$$

To approximate variances, note from the definition (4.4) of a_t^* that its limiting value is

TABLE 3. Means and variances of the number of inspections (N_T) and the number of adjustments (M_T) for three plans listed in Table 2.

Plan	d_1	d_2	$E[N_T]$	$V[N_T]$	$E[M_T]$	$V[M_T]$
II	9	1	155.97	104.83	7.80	7.12
IV	8	3	157.10	71.39	7.86	7.05
VI	7	5	158.30	55.25	7.92	7.08

$$a^*_\infty = (1-q)[p_1^{(\infty)} + \sum_{j=1}^{m} p^{(\infty)}_{\delta+A(j-1)+1}]$$
$$+ \sum_{j=1}^{m} q_j [p_1^{(\infty)} + \sum_{k=1}^{m} p^{(\infty)}_{\delta+A(k-1)+1}] \tag{5.5}$$

i.e., $a^*_\infty = a_\infty$. For some specified error $\varepsilon > 0$ let $\tau_1 = \min\{t \,|\, |a_t - a_\infty| < \varepsilon\}$, $\tau_2 = \min\{t \,|\, |a^*_t - a_\infty| < \varepsilon\}$, and $\tau = \max\{\tau_1, \tau_2\}$. Following Smeach and Jernigan (1977), we compute

$$V(N_{t+1}) - V(N_t) = 2 \sum_{i=1}^{t} (a_i a^*_{|t+1-i|} - a_i a_{t+1}) + a_{t+1} - a^2_{t+1} . \tag{5.6}$$

Because $a_{t+1} \to a_\infty$ and $a^*_u \to a_\infty$, for $t > \tau$ we have the approximation

$$V(N_{t+1}) - V(N_t) \doteq a_\infty[1-(2\tau+1)\, a_\infty + 2 \sum_{i=1}^{\tau} a^*_i] . \tag{5.7}$$

It follows that

$$V(N_T) \doteq Ta_\infty[1-(2\tau+1)a_\infty + 2 \sum_{i=1}^{\tau} a^*_i] . \tag{5.8}$$

To compare the approximation with the exact computation, we return to the three sampling plans given in Table 3. Rather than fixing ε arbitrarily, we specify $\tau = 100$ and apply the approximation (5.8) with $T = 1000$. The results are reported in Table 4, suggesting that considerably more than 100 terms would be required for these cases in order to achieve acceptable accuracy.

6. Concluding Remarks. Several variations of our basic model can be given. If no rejection is specified on the range of the quality characteristics $\underset{\sim}{Y}$, we obtain the sampling structure considered in Arnold (1970), Crigler (1973), and Smeach and Jernigan (1977). The sampling states are $\{E_1, \ldots, E_\delta\}$; nonzero elements of the matrix $\underset{\sim}{P}(\delta\times\delta)$ of one-step transition probabilities are $\{P_{i,1};\ i = 1, 2, \ldots, \delta\}$ and $\{P_{i,i+1};\ i = 1, 2, \ldots, \delta-1\}$ with $P_{\delta,1}=1$; and the expected sample size in the interval $(0,T]$ is

TABLE 4. Exact and approximate variance of the number of inspections using τ = 100 terms for three sampling plans listed in Table 2.

Plan	d_1	d_2	$V[N_T]$	Approximation
II	9	1	104.83	144.09
IV	8	3	71.39	109.44
VI	7	5	55.25	92.53

$$E(N_T) = \sum_{t=1}^{T} P_{11}^{(t)} \quad . \tag{6.1}$$

Another interesting possibility is to terminate sampling when $\underset{\sim}{Y} \varepsilon R_j$ for any $j = 1, 2, \ldots, m$. The sampling states are $\{E_1, \ldots, E_\delta, F_1, \ldots, F_m\}$ where F_j is the absorbing state corresponding to $\underset{\sim}{Y} \varepsilon R_j$. Nonzero elements of the matrix of one-step transition probabilities are $\{P_{i,1};\ i = 1, 2, \ldots, \delta\}$, $\{P_{i,i+1};\ i = 1, 2, \ldots, \delta-1\}$, $\{P_{i,\delta+j};\ i = 1, 2, \ldots, \delta,\ j = 1, 2, \ldots, m\}$, and $\{P_{\delta+j,\delta+j}=1;\ j = 1, 2, \ldots, m\}$. This strategy for monitoring production is analogous to the cumulative result criterion of Cone and Dodge (1964) in lot inspection, where sampling is terminated when the cumulative result from a starting point exceeds its criterion.

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Sampling procedures are considered for monitoring the output of a production process. Sampling rates are allowed to depend on one of several levels of acceptance or rejection, sampling less frequently when the process is in control. The problem is formulated as a simple Markov process whose

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properties yield the expected values and the variances of the number of samples and the numbers of various types of adjustments to the process. Computations are given in support of the economic design of variable sampling policies of this type.